PHYSICAL REVIEW E VOLUME 60, NUMBER 4 OCTOBER 1999

Thermal noise can facilitate energy conversion by a ratchet system

Fumiko Takagi* and Tsuyoshi Hondou†
Department of Physics, Tohoku University, Sendai 980-8578, Japan

(Received 26 March 1999)

Molecular motors in biological systems are expected to use ambient fluctuation. In a recent paper [Phys. Rev. Lett. **80**, 5251 (1998)], it was shown that the following question was unanswered: Can thermal noise facilitate energy conversion by ratchet system? We consider it using stochastic energetics, and show that there exist systems where thermal noise helps the energy conversion. [S1063-651X(99)01110-1]

PACS number(s): 05.40.Ca, 87.10.+e

Molecular motors in biological systems are known to operate efficiently [1–4]. They convert molecular scale chemical energy into macroscopic mechanical work with high efficiency in water at room temperature, where the effect of thermal fluctuation is unavoidable. These experimental facts lead us to anticipate the existence of the system where thermal noise helps motor operation. Determining the mechanism of these motors is useful not only in biology but also in statistical and thermal physics.

Recently inspired by observations of the molecular motors, many studies have been performed from the viewpoint of statistical physics. Much has been studied in ratchet models [5–7] to determine how the directed motion emerges out of nonequilibrium fluctuation. One of the best known works among these ratchet models was by Magnasco [8]. He studied the ‘‘forced thermal ratchet,’’ and claimed that ‘‘there is a region of the operating regime where the efficiency is optimized at finite temperatures.’’ His claim is interesting because thermal noise is known to usually disturb the operation of machines. However, it was recently revealed that this claim was made incorrectly [9], because it was not based on analysis of the energetic efficiency but only on that of the probability current, as most of the studies of ratchet systems were. The insufficient analysis was attributed to the lack of systematic methods of studying energetics in systems described by the Langevin equation. Recently, a method called stochastic energetics was formalized, where the heat was described quantitatively in the framework of the Langevin equation [10]. Using this method, some attempts to discuss the energetics of these systems [11–14] have been made. Through the energetic formulation of the forced thermal ratchet [9] using this stochastic energetics, the following was shown: The behavior of the probability current is qualitatively different from that of energetic efficiency. Thermal noise does *not* contribute to energy conversion by the ratcheting, at least under the conditions in which the claim was made.

Therefore, it was revealed that the following question had not yet been answered: Can thermal noise facilitate operation of the ratchet? In this paper, we will show that thermal noise certainly can facilitate the operation of the ratchet.

Let us consider an overdamped particle in an ‘‘oscillating ratchet,’’ where the amplitude of the one-dimensional (1D) ratchet potential is constant, but the degree of symmetry breaking oscillates at frequency ω (Fig. 1). The Langevin equation is as follows:

$$\frac{dx}{dt}=-\frac{\partial V(x,t)}{\partial x}+\xi(t), \tag{1}$$

$$V(x,t)=V_p(x,t)+lx, \tag{2}$$

where x, l, and $V_p(x,t)$ represent the state of the system, the load, and the ratchet potential, respectively (Fig. 2). The white and Gaussian random forces $\xi(t)$ satisfy $\langle \xi(t)\rangle=0$ and $\langle \xi(t)\xi(t')\rangle=2\epsilon\delta(t-t')$, where the angular brackets $\langle\,\rangle$ denote the ensemble average. We use the unit $m=\gamma=1$. We assume that the potential $V(x,t)$ always has basins and thus a particle cannot move over the potential peak without thermal noise. The ratchet $V_p(x,t)$ is assumed to satisfy the temporally and spatially periodic conditions

$$V_p(x,t+T)=V_p(x,t), \tag{3}$$

$$V_p(x+L,t)=V_p(x,t), \tag{4}$$

where L is a spatial period of the ratchet potential, and $T(\equiv 2\pi/\omega)$ is a temporal period of the potential modulation. Through potential modulation, energy is introduced into the system and the system converts it into work against the load [15].

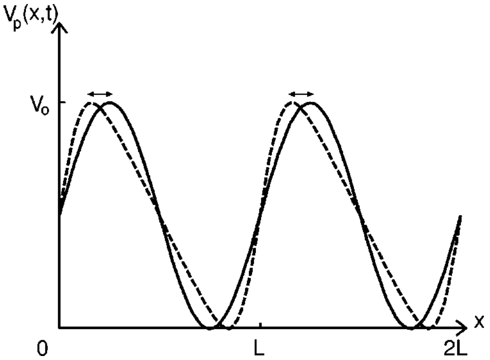

FIG. 1. Oscillating ratchet potential $V_p(x,t)$. The ratchet potential changes continuously between solid line and broken line with the time period T. The amplitude of the ratchet keeps constant, V_0.

---

*Electronic address: fumiko@cmpt.phys.tohoku.ac.jp

†Electronic address: hondou@cmpt.phys.tohoku.ac.jp

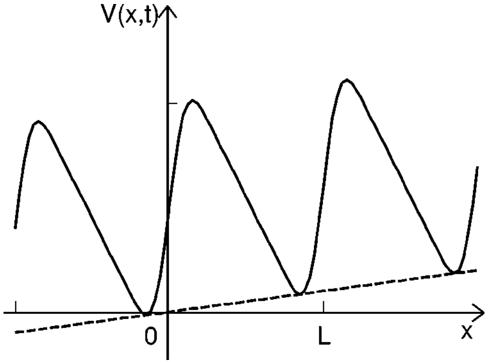

FIG. 2. Snapshot of the potential $V(x,t)$ (solid line). The broken line represents the load term lx.

The Fokker-Planck equation [16] corresponding to Eq. (1) is written

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x},$$

$$= -\frac{\partial}{\partial x}\left(-\frac{\partial V(x,t)}{\partial x} P(x,t) \right) + \epsilon \frac{\partial^2 P(x,t)}{\partial x^2}, \qquad (5)$$

where $P(x,t)$ and $J(x,t)$ are a probability density and a probability current, respectively. We apply the periodic boundary conditions on $P(x,t)$ and $J(x,t)$,

$$P(x+L,t) = P(x,t), \qquad (6)$$

$$J(x+L,t) = J(x,t), \qquad (7)$$

where $P(x,t)$ is normalized in the spatial period L. Except for transient time, $P(x,t)$ and $J(x,t)$ satisfy the temporally periodic conditions

$$P(x,t+T) = P(x,t), \qquad (8)$$

$$J(x,t+T) = J(x,t). \qquad (9)$$

According to the stochastic energetics [10], the heat $\widetilde{Q}$ released to the heat bath during the period T is given as

$$\widetilde{Q} = \int_{x(0)}^{x(T)} \left\{ -\left(-\frac{dx(t)}{dt} + \xi(t) \right) \right\} dx(t). \qquad (10)$$

Inserting Eq. (1) into Eq. (10), we obtain the energy balance equation

$$\widetilde{Q} = \int_0^T \frac{\partial V(x(t),t)}{\partial t} dt - \int_{V(0)}^{V(T)} dV(x(t),t). \qquad (11)$$

The first term on the right-hand side is the energy $\widetilde{E_{\rm in}}$ that the system obtains through the potential modulation, and the second term, $\int_{V(0)}^{V(T)} dV(x(t),t)$, is the work $\widetilde{W}$ that the system extracts from the input energy $\widetilde{E_{\rm in}}$ during the period T. The ensemble average of $\widetilde{W}$ is given using Eqs. (2), (3), and (8) as

$$\langle \widetilde{W} \rangle = \left\langle \int_{V(0)}^{V(T)} dV(x(t),t) \right\rangle = l \int_0^T dt \int_0^L dx J(x,t) \equiv W, \qquad (12)$$

where W represents the work against the load. Also, using Eqs. (2), (5), and the periodic conditions [Eqs. (3), (4), (7), and (8)], the ensemble average of $E_{\rm in}$ is given as

$$\langle \widetilde{E_{\rm in}} \rangle = \left\langle \int_0^T \frac{\partial V(x(t),t)}{\partial t} dt \right\rangle$$

$$= \int_0^T dt \int_0^L dx \left(-\frac{\partial V_p(x,t)}{\partial x} \right) J(x,t) \equiv E_{\rm in}. \qquad (13)$$

Therefore, we obtain the efficiency η of the energy conversion from the input energy $E_{\rm in}$ into the work W as follows:

$$\eta = \frac{W}{E_{\rm in}} = \frac{l \int_0^T dt \int_0^L dx J(x,t)}{\int_0^T dt \int_0^L dx \{ -[\partial V_p(x,t)]/\partial x \} J(x,t)}. \qquad (14)$$

This expression can be estimated simply by solving the Fokker-Planck equation [Eq. (5)].

We solve Eq. (5) numerically with the following ratchet potential as an example. It satisfies Eqs. (3), (4), and the condition that the degree of the asymmetry oscillates but the amplitude of the ratchet is constant. It will turn out that the result does not depend on the detailed shape of the potential. The ratchet potential is

$$V_p(x,t) = \frac{1}{2} V_0 \left(\sin\left\{ \frac{2\pi x}{L} + A(t) \sin \left[\frac{2\pi x}{L} + C_1 \sin\left(\frac{2\pi x}{L} \right) \right] \right\} + 1 \right), \qquad (15)$$

where $A(t) = C_2 + C_3 \sin(\omega t)$ and V_0, C_1, C_2, C_3 are constant.

The results are shown in Fig. 3. We find that the efficiency is maximized at finite intensity of thermal noise [Fig. 3(a)]. This shows that thermal noise can certainly facilitate the energy conversion. What is the reason for the behavior of the efficiency η? Let us regard the work W and the input energy $E_{\rm in}$ as a function of the intensity of thermal noise. The work W, the numerator of Eq. (14), has a peak at finite intensity of thermal noise [Fig. 3(b)] because of the stochastic-resonance-like behavior of the flow during the period T, $\overline{J} \equiv \int_0^T dt \int_0^L dx J$. In the absence of thermal noise ($\epsilon = 0$), the particle cannot move over the potential peak (which results in $\overline{J} = 0$). As the intensity of thermal noise increases, the effect of nonequilibility emerges and it induces finite asymmetric flow against the load through the asymmetry of the ratchet. When thermal noise is large enough ($\epsilon \to \infty$), the flow against load is no longer positive, because the effect of the ratchet disappears in this limit. Therefore, the flow, and also the work, behave like Fig. 3(b) as a function of thermal noise intensity. The input energy $E_{\rm in}$, the denominator of Eq. (14), remains finite at the limit $\epsilon \to 0$ [Fig. 3(c)], where all input energy dissipates because the oscillation of the local potential minimum makes finite local current even in the absence of thermal noise. Therefore, the efficiency begins with $\eta = 0$ at $\epsilon = 0$ and increases as the

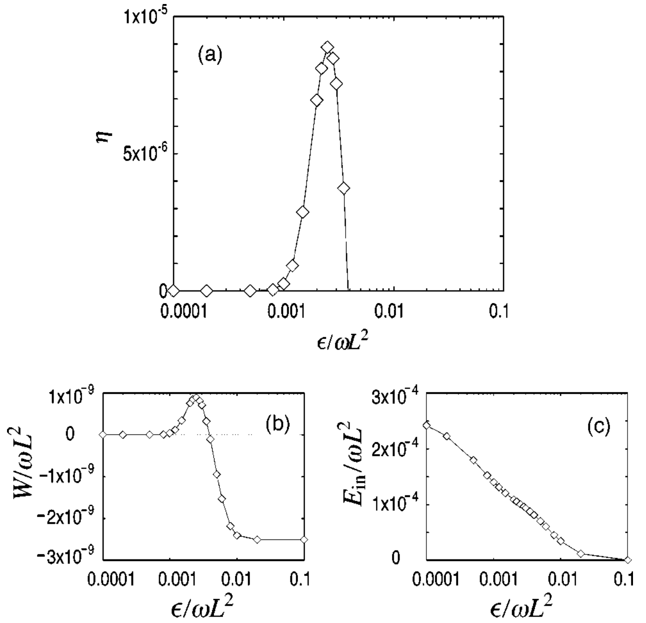

FIG. 3. Energetic efficiency, $\eta = W/E_{\rm in}$, of the oscillating ratchet system as a function of thermal noise intensity, where $V_0/\omega L^2 = 0.01$, $l/\omega L = 0.00002$, $C_1 = 0.3$, $C_2 = 0.3$, and $C_3 = 0.3$. (a) Efficiency η, (b) work W, and (c) input energy $E_{\rm in}$.

intensity of thermal noise increases, then disappears as $\epsilon \to \infty$. The efficiency has its peak at finite ϵ.

As we have stated above, noise-induced flow and finite dissipation in the absence of thermal noise are the cause of the noise-induced energy conversion. Thus our finding will not depend on the detail of the shape of $V_p(x,t)$. We expect that thermal noise can facilitate the energy conversion in a variety of ratchet systems.

Finally, we discuss the forced thermal ratchet [8]. The forced thermal ratchet is a system where a dissipative particle in a ratchet is subjected to both zero-mean external force and thermal noise. The previous paper [9] presented the first trial that discussed the energetics in the ratchet. For the analytical estimate, the discussion in that paper was only on the quasistatic limit where the change of the external force is slow enough. In that case, thermal noise cannot facilitate operation of the ratchet. The energetic efficiency is a monotonically decreasing function of thermal noise intensity, in contrast to the oscillating ratchet discussed above. However, one can see that the external force of the forced thermal ratchet can also be written by an oscillatory modulating potential, when the external force is periodic as in the literature [8,9]. It is likely that the difference between the two cases, the oscillating ratchet and the forced thermal ratchet discussed in that paper [9], is attributable to the different conditions of the two systems, namely, one is quasistatic and the other is not. Thus, we suppose that thermal noise may facilitate the energy conversion in the forced thermal ratchet when the ratchet is not quasistatic.

The Langevin equation of the forced thermal ratchet is the same as Eq. (1), except for the potential V. In this case, the potential is

$$V(x,t) = V_p(x) + lx - F_{\rm ex}(t)x, \tag{16}$$

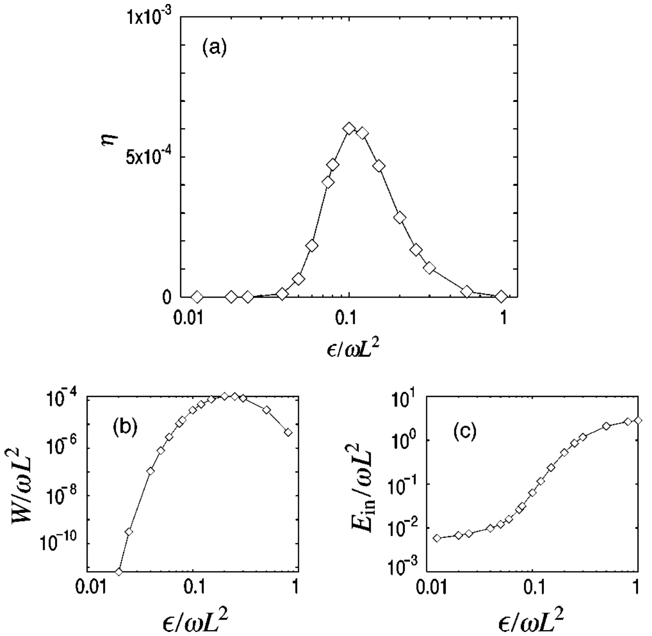

FIG. 4. Energetic efficiency, $\eta = W/E_{\rm in}$, of the forced thermal ratchet as a function of thermal noise intensity, where $V_0/\omega L^2 = 1.0$, $l/\omega L = 0.001$, and $|F_{\rm ex}|_{\rm max}/\omega L = 1.0$. (a) Efficiency η, (b) work W, and (c) input energy $E_{\rm in}$.

where $V_p(x)$, l, and $F_{\rm ex}$ represent the ratchet potential, load, and an external force, respectively. The periodic external force $F_{\rm ex}(t)$ satisfies $F_{\rm ex}(t+T) = F_{\rm ex}(t)$ and $\int_0^T dt F_{\rm ex}(t) = 0$ [17]. The work W is the same as Eq. (13), and the input energy $E_{\rm in}$ is

$$E_{\rm in} = \int_0^T dt \int_0^L dx F_{\rm ex}(t) J(x,t). \tag{17}$$

In the quasistatic limit [9], the probability current J does not depend on the coordinate x. Thus, when the current over the potential peak (that causes W) vanishes, the local current vanishes everywhere $[J(x,t) = J(t) = 0]$. However, if the system is not quasistatic, the behavior changes qualitatively (Fig. 4). In this case, even when the current over the potential peak vanishes at $\epsilon = 0$, the local current around the local potential minimum still remains finite. Thus there exists finite energy dissipation even in the limit $\epsilon \to 0$, which means that the input energy $E_{\rm in}$ still remains a finite value at this limit [Fig. 4(c)]. Therefore, the efficiency is found to be zero at $\epsilon = 0$, and has a peak at finite ϵ [Fig. 4(a)]. The result is the same as that of the oscillating ratchet. It must be noted that the energetics can distinguish the behavior of the efficiency in the nonquasistatic case from that in the quasistatic case, although the dependences of the flow $\overline{J}$ are the same in both.

We have discussed energetics of the ratchet system using the method of the stochastic energetics, and estimated the efficiency of energy conversion. We found that thermal noise *can* facilitate the operation of the ratchet system. The mechanism was briefly summarized as follows. Through the ratchet, potential modulation causes noise-induced flow against the load that results in the work. On the other hand, potential modulation with finite speed causes a local current around the local potential minimum that causes finite dissipation even in the absence of thermal noise. Thus the efficiency is maximized at finite intensity of thermal noise. The result must be robust and independent of the detail of the potential because only two factors are essential for the energy conversion activated by thermal noise: one is the noise-induced flow, and the other is the finite dissipation in the absence of thermal noise. Also in the two-state model [7], another type of ratchet system, it was reported quite recently that the efficiency could be maximized at finite temperature [18]. Whether and how the real molecular motors use thermal noise is a subject for future experimentation.

We would like to thank K. Sekimoto, J. Prost, A. Parmeggiani, F. Jülicher, S. Sasa, T. Fujieda, and T. Tsuzuki for helpful comments. This work was supported by the Japanese Grant-in-Aid for Science Research Fund from the Ministry of Education, Science and Culture (No. 09740301) and the Inoue Foundation for Science.

---

[1] T. Yanagida, T. Arata, and F. Oosawa, Nature (London) **316**, 366 (1985).
[2] A. Ishijima, H. Kojima, H. Higuchi, Y. Harada, T. Funatsu, and T. Yanagida, Biophys. J. **70**, 383 (1996).
[3] T. Q. P. Uyeda, S. J. Kron, and J. A. Spudich, J. Mol. Biol. **214**, 699 (1990).
[4] R. Yasuda, H. Noji, K. Kinosita, F. Motojima, and M. Yoshida, J. Bioenerg. Biomembr. **29**, 207 (1997).
[5] R. D. Vale and F. Oosawa, Adv. Math. **26**, 97 (1990).
[6] A. F. Huxley and R. M. Simmons, Nature (London) **233**, 533 (1971).
[7] See, e.g., F. Jülicher, A. Ajdari, and J. Prost, Rev. Mod. Phys. **69**, 1269 (1997), and references therein.
[8] M. O. Magnasco, Phys. Rev. Lett. **71**, 1477 (1993).
[9] H. Kamegawa, T. Hondou, and F. Takagi, Phys. Rev. Lett. **80**, 5251 (1998).
[10] K. Sekimoto, J. Phys. Soc. Jpn. **66**, 1234 (1997).
[11] K. Sekimoto and S. Sasa, J. Phys. Soc. Jpn. **66**, 3326 (1997).
[12] M. Matsuo and S. Sasa, Physica A (to be published).
[13] T. Hondou and F. Takagi, J. Phys. Soc. Jpn. **67**, 2974 (1998).
[14] K. Sekimoto, F. Takagi, and T. Hondou, e-print cond-mat/9904322.
[15] In this paper, we discuss the systems that convert mechanical energy into mechanical work, while real molecular motors in biological systems convert chemical energy into mechanical work. Recently, an experiment suggests that the protein can store the chemical energy from ATP hydrolysis [19], the energy of which may be stored in a mechanical way; for example, by conformational change of the protein. Some models have been proposed to explain this kind of energy storage [20].
[16] H. Risken, *The Fokker-Planck Equation*, 2nd ed. (Springer-Verlag Berlin, 1989).
[17] We consider the low amplitude regime [9] where the amplitude of $F_{\rm ex}(t)$ is small. In this case, a particle cannot move over the potential peak without thermal noise, as in the case of the oscillating ratchet.
[18] A. Parmeggiani, F. Jülicher, A. Ajdari, and J. Prost, Phys. Rev. E **60**, 2127 (1999).
[19] A. Ishijima, H. Kojima, T. Funatsu, M. Tokunaga, H. Higuchi, H. Tanaka, and T. Yanagida, Cell **92**, 161 (1998).
[20] See, for example, N. Nakagawa and K. Kaneko, e-print chao-dyn/9903005.